"Numerical Solution of Partial Differential Equation"

References المصادر

- Numerical Solution of partial differential equation by finite differential methods

by G. D. Smith

linear partial differential equation in general

$$A u_{xx} + B u_{yy} + C u_{xy} + D u_x + E u_y + F u = G$$ ← linear

الصيغة العامة للمعادلات التفاضلية الجزئية

المعادلة التفاضلية الجزئية هي عبارة عن معادلة تحتوي على مشتقات جزئية

$\frac{\partial^2 u}{\partial x^2} = u_{xx}$ ← معادلة تفاضلية جزئية من الرتبة الثانية

- الرتبة هي أعلى مشتقة تظهر في المعادلة

$\frac{\partial^2 u}{\partial y^2} = u_{yy}$, $\frac{\partial u}{\partial x} = u_x$,

$\frac{\partial u}{\partial y} = u_y$

A, B, C, D, E and F may be function of x and y.

- اذا كانت ال A, B, C, D, E, F دوال بالنسبة للـ x وال y فنطلق عليها (ذات معاملات متغيرة) With Varrable the Cofficient

$u_x + u^2 = 6$ ← non linear

Note: We have three types of linear partial differential equation
لدينا ثلاث انواع من ال L.P.D.E ويتحدد ذلك من B^2-4AC

B^2-4AC
- $=0$ (parabolic) ⇝ heat eq. $u_t = u_{xx}$
- >0 (hy parabolic) ⇝ wave eq. $u_{tt} = c^2 u_{xx}$
- <0 (Elliptic) ⇝ Laplace eq. $u_{xx} + u_{yy} = 0$

تصنيف هام ويعتمد عليه بالكورس

Method of Solution

1- Separation of variables method (Linear & homo).

2- Eigenfunction Expansion method.

3- Integral Transforms methods ⟵ Laplace, Fourier

4- Variational methods

5- Numerical methods (Finite difference method)
⟵ هذه الطريقة التي سنشتغل عليها هذا الكورس

Note: أفضل طرائق لحل المعادلات التفاضلية الجزئية هي تلك الطرائق التي تحول المعادلة التفاضلية الجزئية الى معادلة تفاضلية اعتيادية.

Note: المعادلة التفاضلية التي راح نشتغل عليها في هذا الكورس هي معادلة تفاضلية مع شروط ابتدائية initial و شروط حدودية boundary.

Finite difference approximation derivatives

When a function u and it's partial derivatives are Single value, finite and Continuous of x then by Taylor's theorem.

$$u(x+h) = u(x) + h\acute{u}(x) + \frac{h^2}{2}\acute{\acute{u}}(x) + \frac{1}{6}h^3\acute{\acute{\acute{u}}}(x) + \dots \quad \text{---(1)}$$

$$u(x-h) = u(x) - h\acute{u}(x) + \frac{h^2}{2}\acute{\acute{u}}(x) - \frac{1}{6}h^3\acute{\acute{\acute{u}}}(x) + \dots \quad \text{---(2)}$$

Addition of these equations gives

$$u(x+h) + u(x-h) = 2u(x) + h^2\acute{\acute{u}}(x) + O(h^4) \quad \text{---(3)}$$

$$\acute{\acute{u}} \simeq \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \quad \text{---(4)}$$

↳ Central

$$\acute{u} = \frac{1}{h}\{u(x+h) - u(x)\}$$

Note: الطريقة في finite method ال تحول المعادلة التفاضلية الى معادلة جبرية.

With a leading error in the right hand side of order h^2.

Substraction of equation (2) from equation (1) and nelglect of terms of order h^3 leads to

$$\acute{u}(x) = \frac{\partial u}{\partial x} \simeq \frac{1}{2h}\{u(x+h) - u(x-h)\} \quad \text{---(5)}$$

with an error of order h^2 ↳ Central

هذه الصيغة هي أكثر دقة من الصيغة الأولى من حيث backworld forword لان ال leading error هو h^2.

We can also approximate the first order derivative by the forword difference formula:-

$$\dot{u}(x) \simeq \frac{1}{h}\{u(x+h) - u(x)\} \quad \ldots (6)$$

and the backword difference formula:-

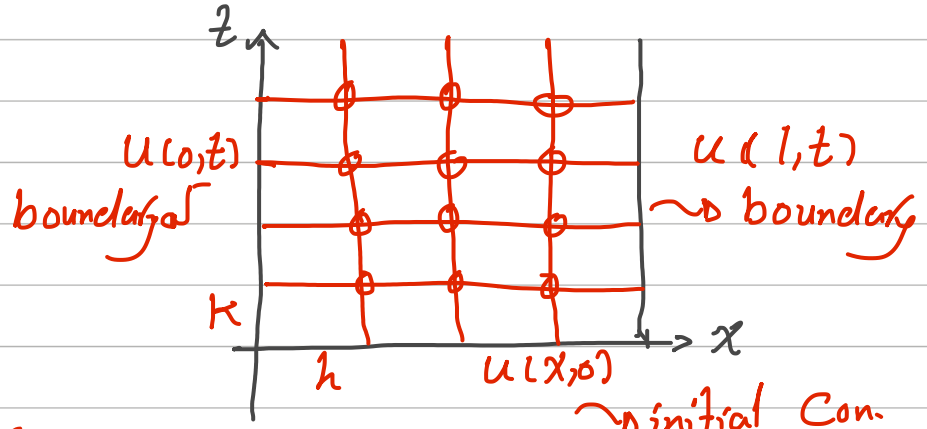
$$\dot{u}(x) \simeq \frac{1}{h}\{u(x) - u(x-h)\} \quad \ldots (7)$$

كلما زاد الأس زادت الدقة

The shows that the leading error in these forword and backword difference formulas are both order (h).

"Notation for function of several variables"

Assume u is a function of the independent variables x and t subdivided the x-t plane into sets of equal rectangles of sides h and k as shown in the following figures:



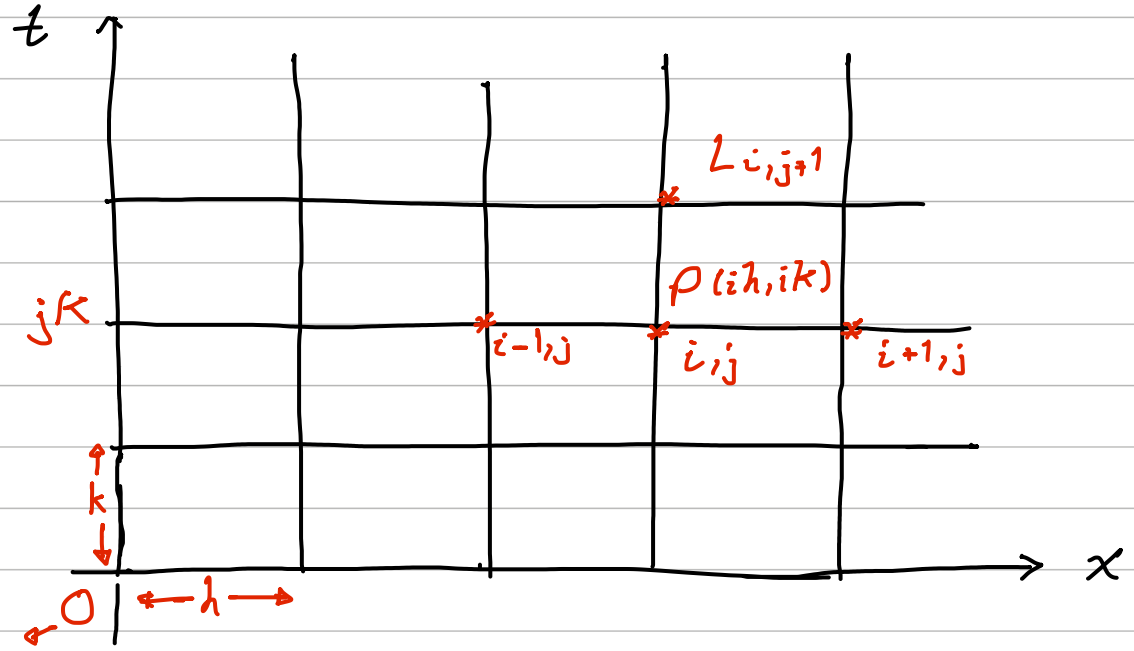
Note: - نقسم x-t إلى شبكة (mesh) ومن
الشبكة تحتوي على مستطيلات والأضلاع متلاقيا h, k

- بما أنو أول الكلام بشكل عمودي فإذا اقسم المنطقة على شكل mesh

- الـ h بمثل على x-axsis و الـ k بمثل على t-axsis

- في الـ x-axsis بين كل نقطتين مسافة ثابتة و هي h .

في الـ t-axsis بين كل نقطتين مسافة ثابتة و هي k .



origen نقطة الأصل

- Note: (Column) العمود i يمثل (row) الصف j يمثل

and let the co-ordinate (x,t) of the representive mesh point P be $x = ih$, $t = jk$ where i and j are integers.

Denote the value of u at P by:

$$u_P(ih, ik) = u_{i,j}$$

by equation (4)

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_P = \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \simeq \frac{u\{(i+1)h, jk\} - 2u\{ih, ik\} + u\{(i-1)h, jk\}}{h^2}$$

$$\text{i.e. } \frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad \ldots (8)$$

With a leading error of order h^2

Note: عند اشتقاق العامل x نعتبر t الثابت
عند اشتقاق العامل t نعتبر x الثابت

توضيح

توضيح المعادلة

$$u(x+h,t) - u(x,t) + u(x-h,t)$$
$$u(ih+h, jk) - u(ih, jk) + u(ih-h, jk)$$
$$u((i+1)h, jk) - u(ih, jk) + u((i-1)h, jk)$$

With this notation the forward difference approximation for $\frac{\partial u}{\partial t}$ at P is

$$\frac{\partial u}{\partial t} \simeq \frac{u_{i,j+1} - u_{i,j}}{k} \quad \cdots (10)$$

Similarly $\frac{\partial^2 u}{\partial t^2} \simeq \frac{u_{i,j+1} - u_{i,j} + u_{i,j-1}}{k^2} \quad \sim (9$

With a leading error of order k^2

Chapter two

"parabolic Equations"

The one dimensional heat equation can be given by

$$u_t = u_{xx}$$

An explicit method of Solution

By equation (10) and (8) on finite difference approximation to $u_t = u_{xx}$ is

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

where $x = ih \quad (i = 0, 1, 2, \dots)$

$t = jk \quad (j = 0, 1, 2, \dots)$

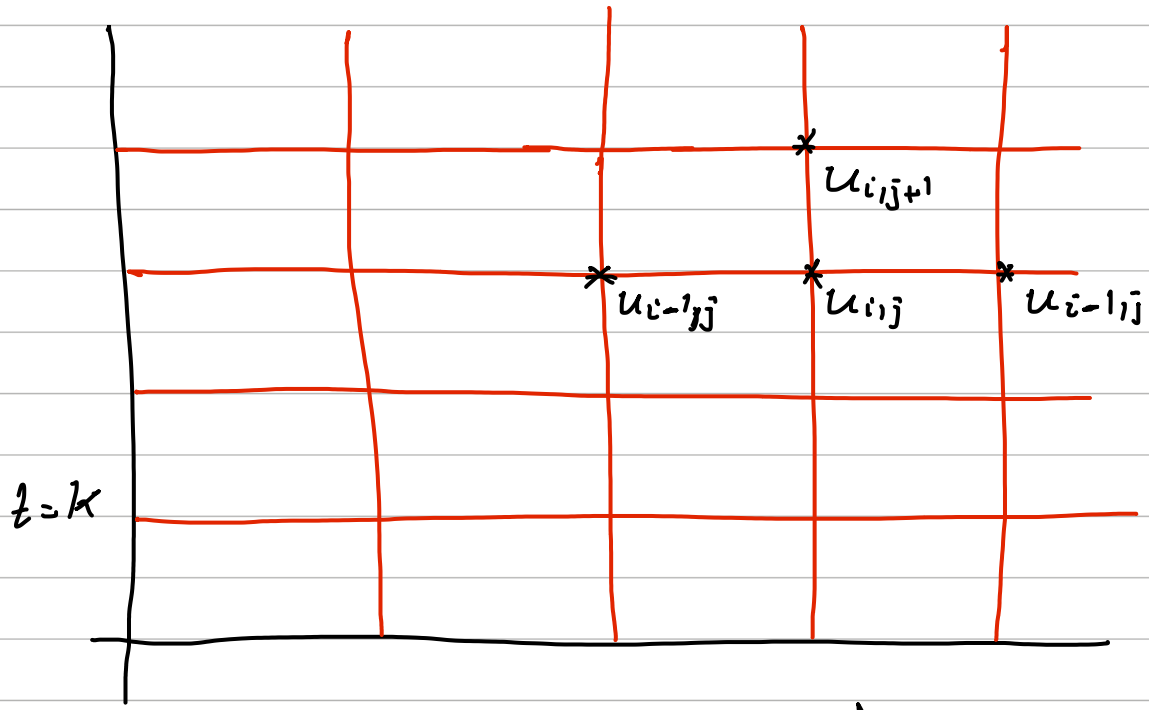
This can be written as

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad \dots (11)$$

where $r = \frac{k}{h^2}$ and gives a formula for the unknown $u_{i,j+1}$ at the $(i, j+1)^{th}$ mesh point in terms of known temperature u along the jth time row as illustraded in the following figure

Note: الصيغ الصريحة يكون المجهول في الطرف الايسر والمعلومات في الطرف الايمن.

- الصيغ الضمنية المجهول في الطرفين



Note: عندما اريد حساب السطر الاول اعتمد على قيم ال initial و عندما اريد حساب السطر الثاني اعتمد على قيم السطر الاول ... الخ .

Hence we can calculate the unknown pivotal values of u along the first time row, $t=k$ in terms of known boundary and initial values along the second time row in terms of the calculated pivotal along the first and so on.

A formula such as this which expresses one unknown pivotal value directly in terms of unknown pivotal value is called explicit formula.

Remark: These three case clearly indicate that the value of r is important and it will be proved lateron that this explicit method is valid only $0 \leq r \leq \frac{1}{2}$

Ex: As a numerical example let us solve

$u_t = u_{xx}$ with

$u = 2x \quad 0 \le x \le \frac{1}{2}$, $u = 2(1-x) \quad \frac{1}{2} \le x \le 1$

and $u = 0$ when $x = 0$ and $x = 1$

using the explicite formula (11).

In other words we are seeking a numerical solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ which satisfies

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}\right\} \text{ Boundary Conditions}$$

and

$$u(x,0) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$ Initial Condition

منا تحسب القيم لـ $0 \le x \le \frac{1}{2}$ لان الي وراها تطلع symmetric

$x=0$	0.1	0.2	0.3	0.4	0.5	0.6	0.8
$j=0$ 0	0.2	0.4	0.6	0.8	1	Symetric	
$j=1$	باقي القيم بالكتاب						

Case 1: Take $h = 0.1$ and $k = 0.001$ So $r = \frac{k}{h^2} = \frac{1}{10}$

$$u_{i,j+1} = \frac{1}{10}(u_{i-1,j} + 8u_{i,j} + u_{i+1,j})$$

Table 2-4

$i=1, j=0$ $= \frac{1}{10}[u_{0,0} + 8u_{1,0} + u_{2,0}]$

$= \frac{1}{10}[0.8 + 8 \times 0.2 + 0.4 = 0.2$

$$u_{2,1} = \frac{1}{10}(u_{1,0} + 8u_{2,0} + u_{3,0})$$

$i=2, j=0$ $= \frac{1}{10}(0.2 + 8 * 0.4 + 0.6) = 0.4$

Case 2: Take $h = \frac{1}{10}$, $k = \frac{5}{1000}$ So $r = \frac{k}{h^2} = 0.5$

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$$

Table 2-5

Case 3: Take $h = 0.1$, $k = 0.01$ So $r = \frac{0.01}{0.01} = 1$

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = u_{i-1} - u_{i,j} + u_{i+1,j}$$

H.W: Calculate table 2-7

Crak - Nicolson implicit method

Although the explicite method is Computationally Simple it has one serious drawback. The time step k نقطة عيب is neccessarily very small because the process is valid only for $0 < \frac{k}{h^2} \leq \frac{1}{2}$, i.e. $k \leq \frac{1}{2} h^2$ and h must be kept small in order to attain resonable accuracy.

Crank and Nicolson (1947) proposed and used. amethod that reduces the total volume of Calculation aned is Valid (i.e. Convergent and stable) for all finite values of r.

They replaced $\frac{\partial^2 u}{\partial x^2}$ by the mean it's finite differences representation on the $(j+1)^{th}$ aned j^{th} time rows and approximated the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

- استبدلوا المشتقة الجزئية الثانية بالنسبة لـ x بأخذ المعدل بالنسبة للصف j+1 و j

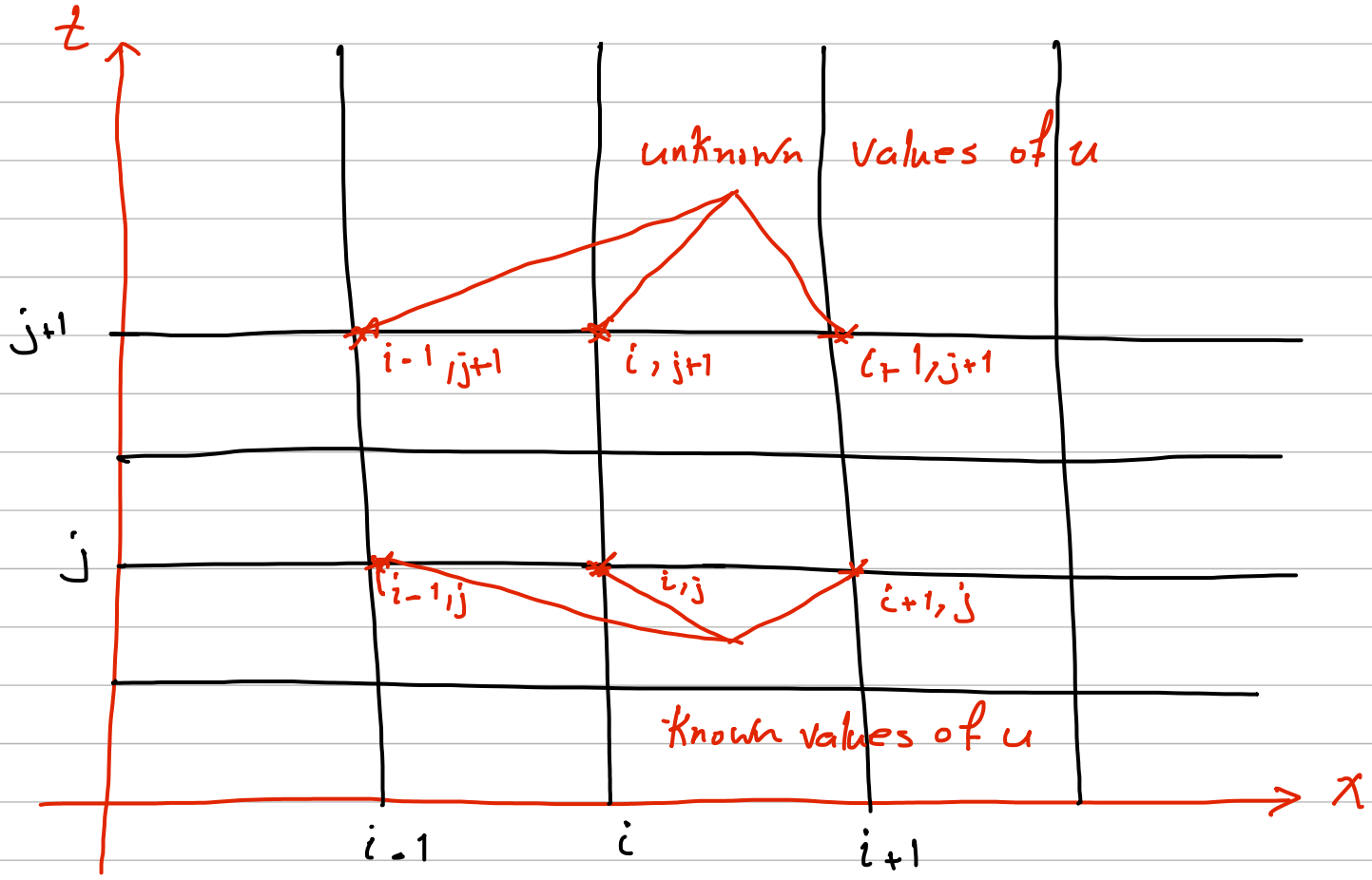
$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{1}{2}\left\{\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}+\frac{u_{i+1,j}-2u_{i,j}-u_{i-1,j}}{h^2}\right\}$$

given

$$-r\,u_{i-1,j+1}+(2+2r)\,u_{i,j+1}-r\,u_{i+1,j+1}=r\,u_{i-1,j}+(2-2r)\,u_{i,j}+r\,u_{i+1,j} \quad \text{---(12)}$$

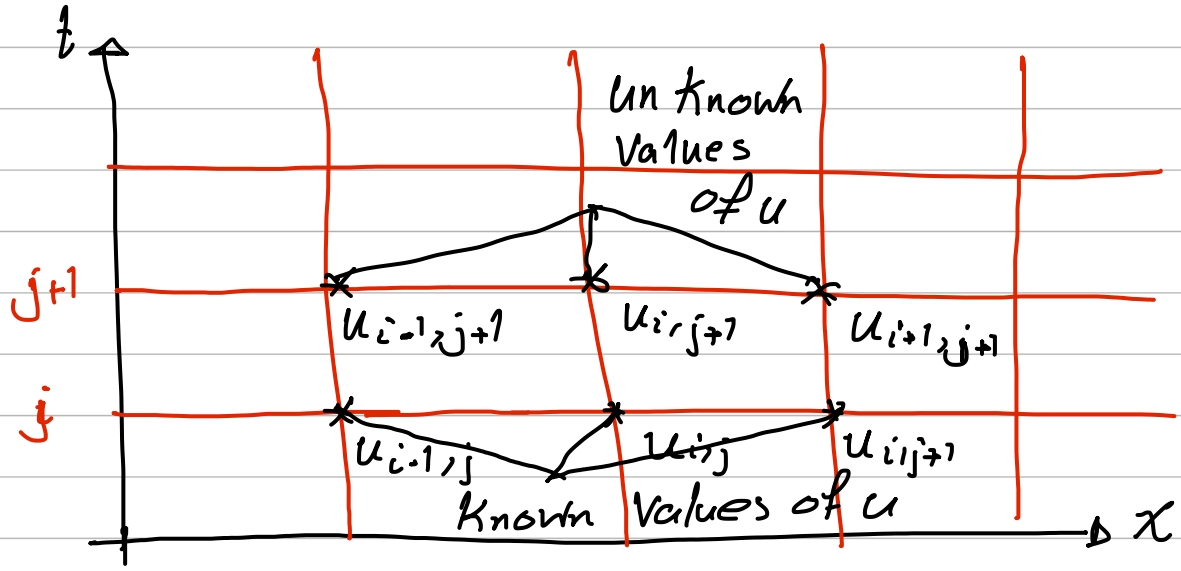
Where $r=\frac{k}{h^2}$

افضل قيمة لاختيار الـ r هي 1/2 لانه كلما قللت حساب الارتام كلما اقتربت الى r=1/2



In general the left side of equation (12) Contains three unknown and the right side three known pivotal values of u as illustrated in the following figure

بصورة عامة في الطرف الايسر لدينا ثلاث مجاهيل وفي الطرف الايمن لدينا ثلاث معاليم



If there are N internal mesh point a long each time row then for $j=0$ and $j=1,2,\dots,N$ equation (12) time gives N simultaneous equation for the unknown pivotal a long the first time row in terms of known initial and boundary values

اذا كان لدينا N من النقاط على طول السطر الي اريد أحسبه فالمعادلة ١٢ تعطينا N من المعادلات المجاهيل الي اريد احسبهم بدلالة ال boundary وال initial

Similarly $j=1$ expresses N unknown values of u a long the second time row in terms of the calculated values a long the first, etc.

عشان اريد حساب السطر الأول اعتمد على ال initial و ال boundary

في حساب السطر الثاني اعتمد على السطر الأول.

A method such as this where the calculation of an unknown pivotal values necessitates the solution of a set simultaneous equation is described as an implicit one

الطريقة التي تكون مجموعة من المعادلات الآنية تسمى implicit

Ex: Use the Crank-Nicolson method to calculate a numerical solution of the previous (سابق) worked example, namely

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1$$

where

$$B.C's = \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad t \geq 0$$

$$I.C \quad u(x,0) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Take $h = 0.1$, Although the method is valid for all finite values of $r = \frac{k}{h^2}$ a large value will yield an inaccurate approximation for $\frac{\partial u}{\partial t}$.

القيمة الكبيرة لـ r تعطي قيم كبيرة وغير دقيقة لتقريب المشتقة u.

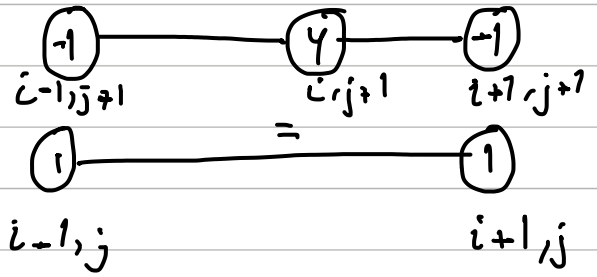
A suitable value is $r = 1$ and has the advantage of making the coefficient of $u_{i,j}$ zero in (12).

القيمة المناسبة لـ r يساوي واحد لديها تجعل معامل الحد في منتصف المعادلة 12 يساوي صفر.

Take $k = 0.01$ and reads as:

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad \text{---(13)}$$

The Computational molecule corresponding to equation (13) is shown in the following figure

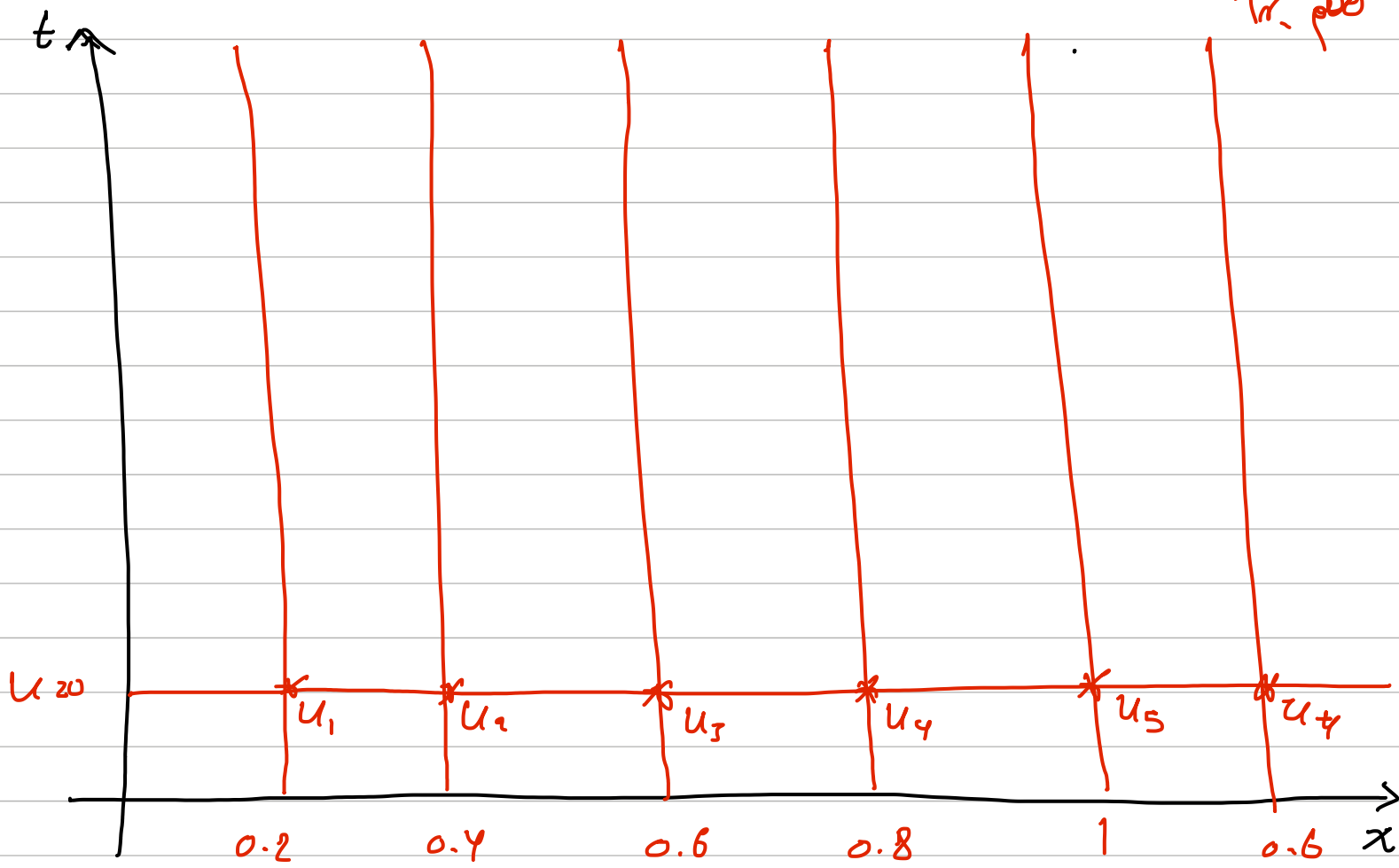


Denote $u_{i,j+1}$ by u_i $(i = 1, \dots, 9)$.

For this problem because of symmetry $u_6 = u_4$,

$u_7 = u_3$, $u_8 = u_2$, --- etc.

as seen in the below figure



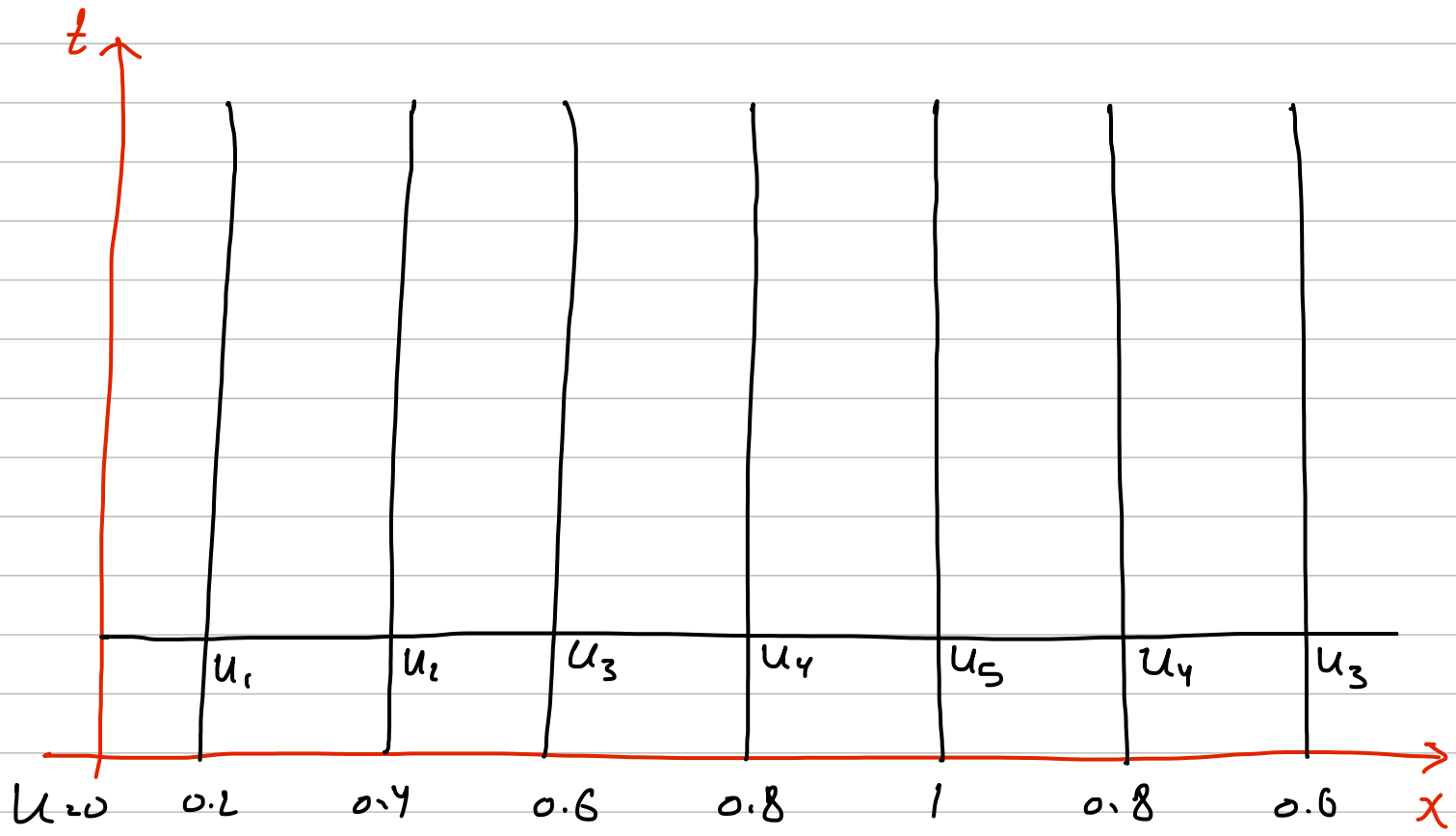
- المجاهيل المجهولة بدلالة u_i والمعاليم بدلالة $u_{i,j}$

The value of u for the first time step then satisfy:

$$-0 + 4u_1 - u_2 = 0 + 0.4$$
$$-u_1 + 4u_2 - u_3 = 0.2 + 0.6$$
$$-u_2 + 4u_3 - u_4 = 0.4 + 0.8$$
$$-u_3 + 4u_4 - u_5 = 0.6 + 1$$
$$-2u_4 + 4u_5 = 0.8 + 0.8$$

$j = 0$
$i = 1, 2, 3, 4, 5$ into equation (3)

$-u_4 + 4u_5 - u_4$ ← because Symmetric



توضيح

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

$$-u_{i-1} + 4u_i - u_{i+1} = u_{i-1,0} + u_{i+1,0}$$

$$-u_0 + 4u_1 - u_2 = 0 + 0.4$$

المجاهيل بدلالة i و المعاليم بدلالة j و i

Note: حيث بعد انتهائك بـ Guass Sidel و Jackobe حيث أن ال System يكون diagonal dominate.

- ال diagonal dominate يؤدي ال System الى non singular يعني يؤدي الى unique sol.

- ال diagonal dominate حاصل x_1 اكبر او يساوي من مجموع الـ Coffecient

- ال Strictly diagonal dominate حاصل جمع المطلق من a_{21} الى a_{2n} كلهم اقل من a_{22}.

An indicated later on these are easily solved by systematic elimination to give $u_1 = 0.1989$,

$u_2 = 0.3956$, $u_3 = 0.5834$, $u_4 = 0.7381$, $u_5 = 0.7691$

Hence the equations for the pivotal values of u along the next time row are:-

$$-0 + 4u_1 - u_2 = 0 + 0.3956$$
$$-u_1 + 4u_2 - u_3 = 0.1989 + 0.5834$$
$$-u_2 + 4u_3 - u_4 = 0.3956 + 0.7381$$
$$-u_3 + 4u_4 - u_5 = 0.5834 + 0.7691$$
$$-2u_4 + 4u_5 = 2 * 0.7381$$

the solution of these equations is given in table 2.8 p. 22

H.W: p. 23

Solution the equations by Gauss's elimination method (Without pivoting).

Derivaitive boundary Condition

Example : Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0<x<1 \qquad \text{(14)}$$

Satisfing the initial Conditions $u=1$ for $0 \le x \le 1$ when $t=0$ and the boundary Conditions

$$\frac{\partial u}{\partial x} = u \quad \text{at} \quad x=0 \quad \text{for all } t$$

هنا لدي قيم مشتقة ال u

$$\frac{\partial u}{\partial x} = -u \quad \text{at} \quad x=1 \quad \text{for all } t$$

using an explicit method and employing Central differences for the boundary Condition.

هنا لدينا قيمة المشتقة

$$\frac{\partial u}{\partial x} = u \quad \text{at} \quad x=0 \quad \text{for all } t$$

$$\frac{\partial u}{\partial x} = -u \quad \text{at} \quad x=1 \quad \text{for all } t$$

Using an explicit method and employing Central differences for the boundary Condition.

one explicit finite difference representation of equation (14) is

$$u_{i,j+1} = u_{i,j} + r\,(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \qquad \text{(15)}$$

Where

$$r = \frac{k}{h^2}$$

في هذه المسألة أختلاف في قيم ال boundary

At $x=0$

$$u_{0,j+1} = u_{0,j} + r\,(u_{-1,j} - 2u_{0,j} + u_{1,j}) \qquad \text{(16)}$$

هنا حصلت عندي مشكلة وهي فكرة $u_{-1,j}$ أحد الحلول اعتمد على حيلة الـ finite difference الي موجودة بال boundary.

The boundary condition at $x=0$, in terms of central difference can be written as

$$\frac{u_{1,j} - u_{-1,j}}{2h} = u_{0,j} \qquad i=0 \qquad \text{---} (17)$$

ضرب طرفين في وسطين

$$\frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

Eliminating $u_{-1,j}$ between (16) and (17) gives

$$u_{0,j+1} = u_{0,j} + 2r\{u_{1,j} - (1+h)u_{0,j}\} \quad \text{---} (18)$$

let $h=0.1$, then at $x=1$ equation (15) becomes

$$u_{10,j+1} = u_{10,j} + r(u_{9,j} - 2u_{10,j} + u_{11,j}) \quad \text{---} (19)$$

and the boundary condition is

$$\frac{u_{11,j} - u_{10,j}}{2h} = u_{10,j} \quad \text{---} (20)$$

Elimination of the fictitious value $u_{11,j}$ between (19) and (20) yields:

$$u_{10,j} = u_{10,j} + 2r\{u_{9,j} - (1+h)u_{10,j}\} \quad \text{---} (21)$$

* This result could have been deduced from the corresponding equation at $x=0$ because of the symmetry with respect to $x=\frac{1}{2}$.

Choose $r=\frac{1}{4}$, the difference equation (18) and (15) then become

$$u_{0,j+1} = \frac{1}{2}\{0.9\,u_{0,j} + u_{1,j}\}$$

$$u_{i,j+1} = \frac{1}{4}\{u_{i-1,j} + 2u_{ij} + u_{i+1,j}\} \qquad i=1,2,3,4$$

and the use of symmetry rather than equation (2.1 gives:

$$u_{5,j+1} = \frac{1}{4}\{2u_{4,j} + 2u_{5,j}\}$$

As the initial temperature is $u=1$ the value of u at the end of the first time step when $k = \frac{1}{400}$

are $u_{0,1} = \frac{1}{2}\{0.9\, u_{0,0} + u_{1,0}\}$

$$= \frac{1}{2}\{0.9(1) + 1\} = 0.95$$

$$u_{1,1} = \frac{1}{4}\{u_{0,0} + 2u_{1,0} + u_{2,0}\}$$

$$= \frac{1}{4}\{1 + 2 + 1\} = 1 = u_{2,1} = u_{3,1} = u_{4,1} = u_{5,1}$$

and the values at the second time stage are:

$$u_{0,2} = \frac{1}{2}\{0.9\, u_{0,1} + u_{1,1}\}$$

$$= \frac{1}{2}\{0.9 * 0.95 + 1\} = 0.9275$$

$$u_{1,2} = \frac{1}{4}\{u_{0,1} + 2u_{1,1} + u_{2,1}\}$$

$$= \frac{1}{4}\{0.95 + 2 + 1\} = 0.9875$$

$$u_{2,2} = \frac{1}{4}\{u_{1,1} + 2u_{2,1} + u_{3,1}\}$$

$$= 1 = u_{3,2} = u_{4,2} = u_{5,2}$$

p. 32, Table 2-10 H.W

Example: Resolve the previous Worked example using an explicit method and employing a forward difference for the boundary Condition at $x=0$ by equation (15), one explicit finite difference representation of the partial differential equation is

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

Hence for $i=1$

$$u_{1,j+1} = u_{1,j} + r(u_{0,j} - 2u_{1,j} + u_{2,j}) \quad \text{— (22)}$$

The boundary Condition at $x=0$, namely $\frac{\partial u}{\partial x} = u$ in terms of forward difference is

$$\frac{u_{1,j} - u_{0,j}}{h} = u_{0,j} \rightarrow u_{0,j} = \frac{u_{1,j}}{1+h} \quad \text{— (23)}$$

Eliminating $u_{0,j}$ between (22) and (23) gives

$$u_{1,j+1} = \left(1 - 2r + \frac{r}{1+h}\right) u_{1,j} + r u_{2,j} \quad \text{— (24)}$$

This Scheme is valid for $0 \leq r \leq \frac{1}{2}$, but in order to Compare it's Solution With the previous one, put $r = \frac{1}{4}$ and $h = 0.1$ the relevant equation are then

$$u_{1,j+1} = \frac{8}{11} u_{1,j} + \frac{1}{4} u_{2,j}$$

$$u_{0,j+1} = \frac{10}{11} u_{1,j+1}$$

$$u_{1,j+1} = \frac{1}{4}(u_{i-1,j} + 2u_{i,j} + u_{i+1,j}) \qquad i = 2,3,4$$

and

$$u_{5,j+1} = \frac{1}{4}(2u_{4,j} + 2u_{5,j})$$

because of Symmetry. Hw: See table 2_13 p. 35 and table 2_14 p. 36

Note: على ضوء هذا المثال نستنتج انه ليس بالضرورة أن نأخذ السؤال السابق لكنه استخدم ال forword و السؤال السابق استخدم ال Central لكنها أيضاً مقبولة.

Ex: Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Satisfying the initial Condition $u(x,0) = 1$

and the boundary Condition

$$\frac{\partial u}{\partial t} = u \qquad x = 0 \qquad \forall t$$

$$\frac{\partial u}{\partial x} = -t \qquad x = 1 \qquad \forall t$$

By the Crank_Nicolson method.

This method represent $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ by

$$\frac{u_{i,j+1} - u_{ij}}{k} = \frac{1}{2}\left\{\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}\right\}$$

Which can be written as

$$-ru_{i-1,j+1} + (2+2r) - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{ij} + ru_{ij+1} \quad \cdots (25)$$

The Central difference representation of the boundary Condition at $x = 0$ is $\frac{u_{1,j} - u_{-1,j}}{2h} = u_{0,j}$

from which it follows that

$$u_{-1,j} = u_{1,j} - 2h\, u_{0,j}$$

and

$$u_{-1,j+1} = u_{1,j+1} - 2h\, u_{0,j+1}$$

The last two equation enable us to eliminate $u_{i-1,j}$ and $u_{-1,j+1}$ from the equation obtained by putting $i = 0$ in eq. (25)

Remarks:

1. The boundary condition at $x = 1$ can be dealt with in the same way although in the problem it is easier to make use of the symmetry with respect to $x = \frac{1}{2}$ namely $u_{6,j} = u_{4,j}$, $u_{7,j} = u_{3,j}$, $u_{8,j} = u_{2,j}$ and so on.

2. This schem is formally valid for all finite values of r but we must keep it resonably small if we want a close approximation to the solution of the partial differential equation, therefore we shall choose $r = 1$ and $h = 0.1$.

$$r = \frac{k}{h^2} \rightarrow k = rh^2 = 0.01$$

بعد عمليات حسابية بسيطة نتحصل على المعادلة التالية

A small amount of algebra soon shows that the equations for the unknown pivotal values $u_{0,j+1}, u_{1,j+1}, u_{2,j+1}, \ldots$ are

$$2.1 \quad u_{0,j+1} - u_{1,j+1} = -0.1\, u_{0,j} + u_{1,j} \quad \text{حوضنا } i=0$$

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad i = 1,2,3,4$$

$$-u_{4,j+1} + 2u_{5,j+1} = u_{4,j} \quad i = 5 \quad r = 1$$

حالة خاصة من المعادلات السابقة

For the first time step these give

$$2.8u_0 - u_1 = 0.9$$
$$-u_0 + 4u_1 - u_2 = 2$$
$$-u_1 + 4u_2 - u_3 = 2$$
$$-u_2 + 4u_3 - u_4 = 2$$
$$-u_3 + 4u_4 - u_5 = 2$$
$$-u_4 + 2u_5 = 1$$

See table 2.15 p. 36

H.W

1- Two dimensional parabolic equations Alternating derivative implicit method. p. 37

2- The parabolic equation in cylindrical and Spherical polar Coordinates p. 40

Chapter_3.

Convergence stability and Consistancy

Descriptive treatment of Convergence:

Let U represent the exact solution of a partial differential equation with independent variables x and t and let u the exact solution of the difference used to approximate the partial differential equation

الطريقة تكون conv. اذا و فقط اذا كانت مستمرة وملتصقة.

الكلام عن ال conv. في المعادلات التفاضلية الجزئية صعب نوعاً ما لذا نبينه بمثال بسيط و

Def: The finite difference equation is said to be Convergence when u tends a fixed point or long a fixed t-level as h and k both tends to Zero.

توضح التعريف

u الى U تتقارب لكي

conv. يجب تكون U

Def: The difference U-u is called the discretization error or the trancation error.

U هي exact Sol. of P.D.E

u هي exact Sol. of finite diff.

Local truncation error and consistensy

Let $F_{i,j}(U)=0$ represent the difference equation at the $(i,j)^{th}$ mesh point. If u is replaced by U at the mesh points of the difference equation then the value of $F_{i,j}(U)$ is called local trancation error at $(i,j)^{th}$ mesh point. If this tends to Zero as the mesh lengths tends to zero the difference equation is said to be Consistent with the partial difference equation.

Descriptive treatment of stability

Let N will be called the numerical Solution and (U-N) the globel rounding error R.

Remark: The total error at the $(i,j)^{th}$ mesh point is

$$U_{ij} - u_{ij} = (U_{ij} - u_{ij}) + (u_{ij} - u_{ij})$$
$$= \text{truncation error} + \text{global rounding error}$$

Stability تنمو bounded إذا بقيت

Def Stability: The difference equation are stable if the growth of R_{ij} is bounded for all i as j tends to infinity.

صنا الـ j يروح لـ ∞ لدن اني اشتغل على اسطر.

Analytical treatment of Convergence (Direct proof):

Convergence is more difficult to investigate than the Stability, therefore only one example will be considered in detail at this stage.

البسيط — حسب جماعتها او التحقق منها Conv. الـ

Denote the exact solution of the PDE U and the exact solution of finite difference equation by u

Then $e = U - u$

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0<x<1 \qquad \sim (26)$$

الـ initial معلوم

Where U is known for $0<x<1$, when $t=0$ at $x=0$ and 1 when $t \geq 0$.

The Simplest explicit finite difference approximation to (26) is

$$\frac{U_{i,j+1} - U_{i,j}}{k} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} \qquad \dots (27)$$

At the mesh point

$$u_{i,j} = U_{i,j} - e_{i,j}$$
$$u_{i,j+1} = U_{i,j+1} - e_{i,j+1}, \text{ etc.}$$

Substition into (27) leads to :

$$e_{i,j+1} = r e_{i-1,j} + (1-2r) e_{i,j} + r e_{i+1,j} + U_{i,j+1} - U_{i,j} + r(2U_{i,j} - U_{i-1,j} - U_{i+1,j}) \quad \sim (28)$$

By Taylor's theorem

$$U_{i+1,j} = U(x_i + h, t_j) = U_{i,j} + h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\frac{\partial^2 U}{\partial x^2}(x_i + \theta_1 h_1, t_j)$$

$$U_{i-1,j} = U(x_i - h, t_j) = U_{ij} - h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\frac{\partial^2 U}{\partial x^2}(x_i - \theta h, t_j)$$

$$U_{i,j+1} = U(x_i, t_j + k) = U_{ij} + k\frac{\partial U}{\partial t}(x_i, t_j + \theta_3 k)$$

Where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and $0 < \theta_3 < 1$

Substitation into equation (28) gives :

$$e_{i,j+1} = r e_{i-1,j} + (1-2r) e_{i,j} + r e_{i+1,j} + k\left\{\frac{\partial U}{\partial t}(x_i, t_j + \theta_3 k) - \frac{\partial^2 U}{\partial x^2}(x_i + \theta_4 h, t_j)\right\} \quad \sim (29)$$

Where $-1 < \theta_4 < 1$

الحد الاعلى للمقدار M

Let E_j denote the maximum value of $|e_{ij}|$ along the jth time row and M the maximum moduln of the expression in the braces for all i and j.

When $r \leq \frac{1}{2}$, all the cofficients of e in the equation (29) are postive or Zero , So

(29) كل المعاملات في معادلة
يكونون اما موجبات او صفر

$$|e_{i,j+1}| \leq r|e_{i-1,j}| + (1-2r)|e_{i,j}| + r|e_{i,j+1}| + k\mu$$

$$\leq r E_j + (1-2r) E_j + r E_j + k\mu$$

$$E_{j+1} = E_j + k\mu \leq (E_{j-1} + k\mu) + k\mu = E_{j-1} + 2k\mu \quad \text{etc.}$$

$E_j \geq |e_{ij}|$ نوعز بـ

from which it follows that

$$E_j \leq E_0 + jk\mu = t\mu$$

initial الـ error في الـ E_0 ← $\frac{jk}{t}$ ← μ max تمثل

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}$$

توضيح

$$\mu = \frac{\partial U}{\partial t}(x_i, t_j + \underbrace{\theta_3 k}_{=0}) - \frac{\partial^2 U}{\partial x^2}(x_i + \underbrace{\theta_4 h}_{=0}, t_j)$$

توضيح

$h \to 0 \quad k \to 0$

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\Big|_{i,j} = 0$$

$E_j \to 0 \quad e_j \to 0$

$(U-u) \to 0 \rightsquigarrow$ converge

because the initial values of u and U are the same $E_0 = 0$ when $h \to 0$, $k = rh^2$ also tends to zero and μ tends to $\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{i,j}$

Since U is a solution of equation (26) the limiting value of μ and therefore of E_j is zero.

As $|U_{ij} - u_{ij}| \leq E_j$, this proves that u converge to U as $h \to 0$ when $r \leq \frac{1}{2}$ and t is finite.

Remark: When $r > \frac{1}{2}$ it can be shown that the complementary function of the difference equation (29) tends to infinity as $h \to 0$.

Analytical treatment of Stability

There are two standerd ways of investigating the boundedness of the solution of the finite difference equations.

In one we express the equations in matrix form and examine the eigen value of an associated matrix in the other, we use a finite Fourier series.

Remarks:

1- The Fourier series method is the easier of the two in that it requires no knowledge of matrix algebra but is less rigorous because it negleets the bounded Cofficients.

2- let u is the solution of the difference equations Corresponding to know initial values.

let u^* is the solution of the same equation but with perturbed initial values.

The difference $(u - u^*)$ has been called the error e.

u^* ← same error نسميها الـ

- صار نخلي همه اريد قليل من ال error نسميها e حتى اشوف قيم ال boundary تبقى ثابتة لو تزايد خلي حال تزايدت معناها عدم Stability.

Matrix Method

Explicit finite difference scheme :
Consider the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0 < x < 1$$

← heat eq.

Where $U = 0$ at $x = 0$ and 1 $t > 0$ and U is known when $t = 0$.

The explicite finite difference approximation

$$U_{i,j+1} = r U_{i-1,j} + (1-2r) U_{i,j} + r U_{i+1,j} \quad -(30$$

leads to the equations

هنا بلشنا من واحد ونتقي ب 1- لأن حتى ابقى ضمن الحدود ال boundary

$$U_{1,j+1} = 0 + (1-2r) U_{1,j} + r U_{2,j}$$

$$U_{2,j+1} = r U_{1,j} + (1-2r) U_{2,j} + r U_{3,j}$$

$$U_{n-1,j+1} = r U_{n-2,j} + (1-2r) U_{N-1,j} + 0$$

المصفوفة تكون tridiagonal

Where $Nh = 1$ and $U_{0,j} = U_{N,j} = 0$

These can be written as :

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{M-1,j+1} \end{bmatrix} = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ & r & (1-2r) & r & \\ & & \ddots & \ddots & \ddots \\ & & & r & (1-2r) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}$$

or as

$$u_{j+1} = A u_j$$

Where u_j denoted the vector

$$u_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{bmatrix}$$ and

$$A = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ & r & (1-2r) & r & \\ & & \ddots & \ddots & \ddots \\ & & & r & (1-2r) \end{bmatrix}$$

هذه المصفوفة التي استخرج منها القيم الذاتية

Hence $u_j = A u_{j-1} = A(A u_{j-2}) = \cdots A^j u_0$

Where u_0 is the vector of initial values. ← initial القيم

Now suppose we introduce errors at every pivotal values a long $t=0$ and start the computation with the vector of values u_0^* instead of u_0 we shall calculate $u_1^* = A u_0^*$, $u_2^* = A u_1^* = A^2 u_0^*$ and finally

$$u_j^* = A^j u_0^*$$

Define the error value e by $e = u - u^*$

Then

$$e_j = u_j - u_j^* = A^j (u_0 - u_0^*) = A^j e_0$$ ← intial

هنا ال initial حماري مقدار قليل من ال error فالحكم يجي عن طريق المصفوفة A عن طريق e-value اشوف شوكت تروح للصفر وبالتالي احكم شوكت تكون stable فتحيي Stability و Conv. يعني تكون Valid.

Showing that the formula for the propogation of the errors is the same as that for the calculation of u.

Remark: The finite difference scheme will be stable when e_j remains bounded as j inceases indefinitely.

This can always be investigateed by expressing the error vectors in termes of the eigenvectors of A.

Def: A matrix A is said to be non-deficient if it has $(N-1)$ linearly independent eigenvalues λ_s of A are all distinct or A is real anel symmetric.

المتجهات تكون مستقلة اذا كانت القيم الذاتية مختلفة

Assume that A is non-deficient matrix then the these eigenvectors can be used as basis for our $(N-1)$ dimensional vector space anel the error vector e_0, with its $(N-1)$ components can be expressed uniquely as a linear combination of them normely

$$e_0 = \sum_{s=1}^{N-1} c_s v_s$$

Where c_s, $s = 1(1)(N-1)$ are known scalars. The errors along the time-level $t=k$ resulting from the initial error e_0 will be given by: (لو تزداد بمقدار واحد)

$$e_1 = A e_0 = A \sum c_s v_s = \sum c_s A v_s$$

But $A v_s = \lambda_s v_s$ by the definition of the eigenvalue.

Therefore $e_1 = \Sigma c_s \lambda_s v_s$

Similarly

$$e_j = \Sigma c_s \lambda_s^j v_s$$

This shows that the errors will not increase exponentially with j provided the eigenvalue with the largest modulus has modulus less then or equal to unity i.e. $|\lambda| \leq 1$

The matrix A can be written as

$$A = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & r & (1-2r) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \end{bmatrix}}_{\text{Identity matrix}} + r \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

(with the central element of the identity matrix = 0)

i.e. $A = I + r T_{N-1}$

Where T_{N-1} is an $(N-1) \times (N-1)$ matrix which eigenvalue λ_s and eigenvectors v_s.

$$\lambda_s = -4 \sin^2\left(\frac{s\pi}{2M}\right), \quad s = 1, 2, \ldots, M-1$$

$$V_s = \left(\sin\frac{s\pi}{M}, \sin\frac{2s\pi}{M}, \ldots, \sin\frac{(M-1)s\pi}{M}\right)$$

H.W See page 86

Hence the eigenvalue and eigenvectors of A are

$$1 + r\left\{-4 \sin^2\left(\frac{s\pi}{2M}\right)\right\}$$

Therefore the condition for stability of the explicit scheme is

$$\left|1 - 4r \sin^2\left(\frac{s\pi}{2M}\right)\right| \leq 1$$

The only useful inquality in

$$-1 \leq 1 - 4r \sin^2\left(\frac{s\pi}{2M}\right)$$

giving $r \leq \frac{1}{2} \sin^2\left(\frac{s\pi}{2M}\right)$ ← يقع بين -1, 1

proving that the scheme is stable for

$$r \leq \frac{1}{2}$$

#

Crank_Nicolson implicit Scheme

The Crank_Nicolson equation can be written as

$$\begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \\ \\ u_{M-1,j+1} \end{bmatrix} =$$

$$\begin{bmatrix} (2-2r) & r & & & \\ r & (2-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (2-2r) & r \\ & & & r & (2-2r) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ \\ u_{M-1,j} \end{bmatrix}$$

i.e $(2I - rT_{M-1})u_{j+1} = (2I + rT_{M-1})u_j$

Hence $u_{j+1} = (2I - rT_{M-1})^{-1}(2I + rT_{M-1})u_j$

By the previous argument these finite difference equations will be stable when the moduli of the eigenvalues of

$$(2I - rT_{M-1})^{-1}(2I + rT_{M-1}) = A$$

are each less then one, As the eigenValues of T_{N-1} are $-4\sin^2\left(\frac{s\pi}{2N}\right)$, the eigenValues of A are

$$\frac{2-4r\sin^2\left(\frac{s\pi}{2N}\right)}{2+4r\sin\left(\frac{s\pi}{2N}\right)}, \quad s=1,2,\ldots,N-1$$

and these are clearly less then one for all postive values for r proving that the Crank-Nicolson equations have unrestricted Stability.

$\not\not$

Useful theorems on bounds for eigenvalue

Gerschgorin's First Theorem

The largest at the moduli of the eigenvalues of the square matrix A can not exceed the largest sum of the moduli of the elements a long any row or any column.

Proof

let λ_i be an eigenvalue of $N \times N$ matrix A and x_i the corresponding with components $V_1, V_2, \dots, V_n$

Then the equation

$$A x_i = \lambda_i x_i$$

in detail is

$$x = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_s \\ \vdots \\ V_n \end{bmatrix}$$

$$a_{11} V_1 + a_{12} V_2 + \dots + V_{1n} V_n = \lambda_i V_1$$
$$a_{21} V_1 + a_{22} V_2 + \dots + a_{2n} V_n = \lambda_i V_2$$
$$\dots$$
$$a_{n1} V_1 + a_{n2} V_2 + \dots + a_{nn} V_n = \lambda_i V_n$$

let V_s be the largest moduli's of $V_1, \dots, V_n$ select the sth equation and divide (اقسم) by V_s, giving

$$\lambda_i = a_{s1}\left(\frac{V_1}{V_s}\right) + a_{s2}\left(\frac{V_2}{V_s}\right) + \dots + a_{sn}\left(\frac{V_n}{V_s}\right)$$

Therefore $|\lambda_i| \le |a_{s1}| + |a_{s2}| + \dots + |a_{sn}|$

because $\left|\frac{V_i}{V_s}\right| \le 1$, $i = 1, 2, \dots, n$

In particular this holds for $|\lambda_i| = \max |\lambda_s|$ $s = 1, 2, \dots, N$

Since the eigenvalues of the transpose of A are the same of those of A the theorem is also true for Column.

$\ll$

Gerschgorin's Circle theorem or Brauer's theorem

Let ρ_s be the sum of the moduli of the elements along the sth row excluding the diagonal element's a_{ss}. Then each eigenvalue of A lies inside or on the boundary of at least one of the circles. $|\lambda_s - a_{ss}| = \rho_s$

Proof By the previous proof

$$\lambda_i = a_{s1}\left(\frac{v_1}{v_s}\right) + a_{s2}\left(\frac{v_2}{v_s}\right) + \cdots + a_{ss} + \cdots + a_{sn}\left(\frac{v_n}{v_s}\right)$$

Hence

$$|\lambda_i - a_{ss}| = \left|a_{s1}\left(\frac{v_1}{v_s}\right) + \cdots + 0 + \cdots + a_{sn}\left(\frac{v_n}{v_s}\right)\right|$$

$$\leq |a_{s1}| + |a_{s2}| + \cdots + |a_{sn}|$$

$$= \rho_s$$

※

Ex: As an illustrative example Consider the Crank-Nicolson equation

$$(2I - rT_{w-1})u_{j+1} = (2I + rT_{w-1})u_j$$

$$= [4I - (2I - rT_{w-1})]u_j$$

which can be written as

$$Bu_{j+1} = (4I - B)u_j$$

giving

$$u_{j+1} = (4B^{-1} - I)u_j$$

Where

$$B = \begin{bmatrix} (2+2r) & -r & & & \\ -r & (2+2r) & -r & & \\ & \cdots & \cdots & \cdots & \\ & & -r & (2+2r) & -r \\ & & & -r & (2+2r) \end{bmatrix}$$

The finite difference equation will be stable when the modulus of every eigenvalues of $(AB^{-1} - I)$ does not exceed one, that is when

$$\left| \frac{\gamma}{\lambda} - 1 \right| \leq 1$$

where λ is the eigenvalue of B.

This equivalent to $\lambda \geq 2$

For the matrix B. $a_{ss} = (2 + 2r)$, $\max \rho_s = 2r$

So Brauer's theorem leads to

$$|\lambda - 2 - 2r| \leq 2r$$

from which it follows that

$$-2r \leq \lambda - 2 - 2r \leq 2r$$

or $$2 \leq \lambda \leq 2 + 4r$$

proving that the equation are unconditionally stable on $\lambda \geq 2$ for all values of r.

Stability criteria for derivative boundary conditions:

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x^2} \qquad 0 < x < 1$$

and the conditions

$$\frac{\partial u}{\partial x} = h_1 (u - v_1) \quad \text{at} \quad x = 0 \quad t \geq 0$$

$$\frac{\partial u}{\partial x} = -h_2 (u - v_2) \quad \text{at} \quad x = 1 \quad t \geq 0$$

where h_1, h_2, v_1, v_2 are constants $h_1 \geq 0$, $h_2 \geq 0$

When the boundary conditions are approximated by the central difference equations

$$\frac{u_{1,j} - u_{-1,j}}{2h} = h_1 (u_{0,j} - v_1)$$

$$\frac{u_{N+1,j} - u_{N-1,j}}{2h} = -h_2 (u_{N,j} - v_2) \qquad \text{with } h = 1$$

and the differential equation by the explicit scheme

$$u_{i,j+1} = r u_{i-1,j} + (1-2r) u_{i,j} + r u_{i+1,j}$$

elimination of $u_{-1,j}$, $u_{N+1,j}$ leads to the equations

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \\ \vdots \\ u_{N+1,j+1} \end{bmatrix} = \begin{bmatrix} \{1-2r(1+h_1 h)\} & 2r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & 2r & [1-2r(1+h_2 h)] \end{bmatrix} \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{N-1,j} \\ \vdots \\ u_{N+1,j} \end{bmatrix}$$

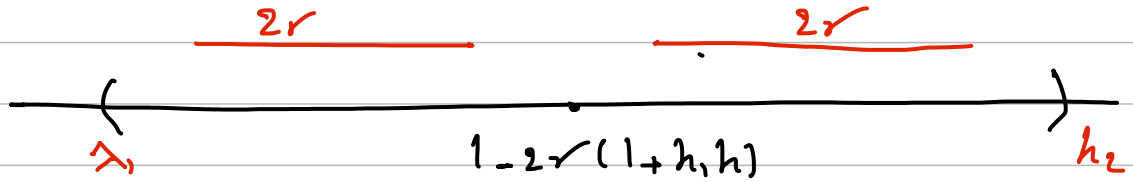
$$+ \begin{bmatrix} 2r h_1 v_1 h \\ 0 \\ 0 \\ \vdots \\ 0 \\ 2r h_2 v_2 h \end{bmatrix}$$

As each component of the last column vector in constant the matrix determing the propogation of the error is

$$\begin{bmatrix} \{1-2r(1+h_1 h)\} & 2r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & 2r & \{1-2r(1+h_2 h)\} \end{bmatrix}$$

Applications of Brauer's theorem to this matrix with $a_{ss} = 1 - 2r(1 + h_1 h)$ and $\rho_s = 2r$

Shows that its eigenvalues λ lies on or with the circle $|\lambda - \underbrace{[1 - 2r(1 + h_1 h)]}_{a_{ss}}| \leq 2r$



$\lambda_1 = 1 - 2r(2 + h_1 h)$

$\lambda_2 = 1 - 2r h_1 h$

and for stability $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$

Hence $-1 \leq 1 - 2r(2 + h_1 h) \leq 1$ giving

$r \leq \frac{1}{2 + h_1 h}$ and $-1 \leq 1 - 2r h_1 h$

giving $r \leq \frac{1}{2 + h_1 h}$

The lest of these is $r \leq \frac{1}{2 + h_1 h}$

Similarly we require $r \leq \frac{1}{2 + h_2 h}$

For row $2(1) \ldots N-1$, $a_{ss} = 1 - 2r$, $\rho_s = 2r$

giving $r \leq \frac{1}{2}$

For over all stability, $r \leq \min\left\{\frac{1}{2 + h_1 h}, \frac{1}{2 + h_2 h}\right\}$

//

Crank - Nicolson equation

It is easily show that the Crank - Nicolson equation for the problem just considered propogate the error by recusion formula

$$e_{j+1} = (\gamma B^{-1} - I)\, e_j$$

where

$$B = \begin{bmatrix} \{2+2r(1+h_1 h)\} & -2r & & & \\ -r & (1+2r) & -r & & \\ & \cdots & \cdots & \cdots & \\ & & -r & (2+2r) & -r \\ & & & -2r & \{2+2r(1+h_2 h)\} \end{bmatrix}$$

As previasuly proved, this scheme is stable when eigenvalues of $B \geq 2$. By Brauers theorem

$$|\lambda - \{2+2r(1+h_1 h)\}| \leq 2r$$

giving $\quad 2+2r h_1 h \leq \lambda$

Similarly the remaining rows give that $2 \leq \lambda$ and $2+2r h_2 h \leq 1$

Hence the equation are unconditionally stable.

$\#\#$

Stability by the Fourier Series method

(Von Neumann's method): p. 93

The Fourier Series can be written be formulation in terms of Sine and Cosines but the algebra is easier if the Complex exponential form is used i.e. With

$\sum a_n \cos\left(\frac{n\pi x}{L}\right)$ or $\sum b_n \sin\left(\frac{n\pi x}{L}\right)$ replaced

by the equivalent $\sum A e^{\frac{in\pi x}{L}}$ where $i^2 = -1$ and L is the initial throughout which the function is defined.

H.W: $e^{\frac{in\pi x}{L}}$ على الصيغة

Clearly we need to chang or usual say in terms of this notation

$$A_n e^{\frac{in\pi x}{L}} = A_n e^{\frac{in\pi ph}{Nh}}$$

$$= A_n e^{iB_n ph}$$

Where

$$B_n = \frac{n\pi}{Nh} \quad \text{and} \quad L = Nh .$$

Denote the error at the pivotal point a long $t=0$, between $x=0$ and NL by $E(ph) = E_p$, $p = 0, 1, \ldots, N$

Then the $(N+1)$ equations

$$E_p = \sum_{n=0}^{N} A_n e^{iB_n ph} \quad (p = 0, 1, \ldots, N)$$

are sufficient to determine the $(N+1)$ unknown $A_0, \ldots, A_n$ uniquely.

Showing that an arbitrary distribution of initial errors can be expressed in this exponential form.

Remarks:

1- The Cofficient A_n is a Constant and Can be neglected.

2- Since our finit difference equations will always be linear and therefore separate solution additive, we need only Consider the propogation of the error due to Single term such as $e^{i\beta p h}$.

To investigate the propogation of this error as t increases we need to find a solution of the finite difference equation which reduces to $e^{i\beta p h}$ when $t = qk = 0$

Assume

$$E_{p,q} = e^{i\beta x} e^{\alpha t} = e^{i\beta p h} e^{\alpha q k}$$

$$\xi = e^{\alpha k} = e^{i\beta p h} \xi^q \quad \cdots (31)$$

and α in general is a Complex Constant.

This obviously reduces to $e^{i\beta p h}$ when $q > 0$.

The error will not increase as t increases provided. $|\xi| \leq 1$

Ex: Investigate the stability of the full implicit finite difference equation

$$\frac{u_{p,q+1} - u_{p,q}}{k} = \frac{u_{p-1,q+1} - 2u_{p,q+1} + u_{p+1,q+1}}{h^2} \quad \text{--- (32)}$$

approximating the perabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Sol

As the error function $E_{p,q}$ satisfies the same difference equation as $u_{p,q}$ substitution $E_{p,q}$ from (31) into equation (29) gives

$$e^{i\beta ph}\xi^{q+1} - e^{i\beta ph}\xi^{q} = r\left\{e^{i\beta(p-1)h}\xi^{q+1} - 2e^{i\beta ph}\xi^{q+1} + e^{i\beta(p+1)h}\xi^{q+1}\right\}$$

Where $r = \frac{k}{h^2}$. Division by $e^{i\beta ph}\xi^{q}$ leads to

$$\xi - 1 = r\xi\left(e^{i\beta h} - 2 + e^{i\beta h}\right)$$

$$= r\xi\left(2\cos\beta h - 2\right)$$

$$= -4r\xi\left(\sin^2(\beta h/2)\right)$$

Hence $$\xi = \frac{1}{1 + 4r\sin^2(\beta h/2)}$$

For stability $|\xi| \le 1$, which is so for all postive value of r.

Ex: The Hyperablic equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is approximated by the explicit scheme

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{h^2} \quad \text{---(33)}$$

Investigate it's stability.

It is easily shown by the procedure of the previous example that the equation for ξ is

$$\xi^2 - 2A\xi + 1 = 0$$

Where $A = 1 - 2r^2 \sin^2(\beta h/2)$, $r = k/h^2$ --- (34)

Hence the values of ξ are

$$\xi_1 = A + (A^2 - 1)^{\frac{1}{2}} \text{ and } \xi_2 = A - (A^2 - 1)^{\frac{1}{2}}$$

For stability

When $A < -1$, $|\xi_2| > 1$ giving in stability

When $-1 \leq A \leq 1$, $A^2 \leq 1$

$$\xi_1 = A + i(1 - A^2)^{\frac{1}{2}}, \quad \xi_2 = A + i(1 - A^2)^{\frac{1}{2}}$$

hence $|\xi_1| = |\xi_2| = \{A^2 + (1 - A^2)\}^{\frac{1}{2}} = 1$

proving that equation (33) is stable for $-1 \leq A \leq 1$, by equation (34) we have $-1 \leq 1 - 2r^2 \sin^2(\beta h/2) \leq 1$

The only useful inequation is

$-1 \leq 1 - 2r \sin^2(\beta h/2)$ giving $r \leq 1$.

Local truncation error

Consider the differential equation

$$L(U_{ij}) = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij} = 0$$

Which is may be replaced exactly at the point (ih, jk) by the forword-difference series:

$$\left(\frac{\partial U}{\partial t}\right)_{ij} = \frac{1}{k}\left(\Delta_t U_{ij} - \frac{1}{2}\Delta_t^2 U_{ij} + \frac{1}{3}\Delta_t^3 U_{ij} + \cdots\right)$$ and the Central Series

$$\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} = \frac{1}{h^2}\left(\sigma_x^2 U_{ij} - \frac{1}{12}\sigma_x^4 U_{ij} + \frac{1}{90}\sigma_x^6 U_{ij} + \cdots\right)$$

To derive a finite-difference equation approximating the differential equation each series must be truncated after a Central number of terms, say for example each series is truncated after the first term.

The approximating difference equation is then the Classical explicit equation

$$F(U_{ij}) = 0 = \frac{1}{k}\Delta_t U_{ij} - \frac{1}{h^2}\sigma_x^2 U_{ij}$$

$$= \frac{U_{i,j+1} - U_{ij}}{k} - \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2} \quad \cdots (35)$$

Remark : The discretization error $U_{ij} - u_{ij}$ denotedls only on h anel K but also on the number of terms in the truncateel series used to approximate each drivative.

The amount by which the exact Solution U of the partial difference equation does not satisfy the difference equation at the point (ih, jk) is Called the local truncation error T_{ij}.

for the scheme defined by equation (35)

$$T_{ij} = F(U_{ij}) = \frac{U_{i,j+1} - U_{i,j}}{K} - \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2} \quad \text{---(36)}$$

This can be written as

$$U_{i,j+1} = K T_{ij} + r U_{i-1,j} + (1-2r) U_{ij} + r U_{i+1,j} \quad \text{---(37)}$$

Ex: Caclulate the order of the local truncation error of the classical explicit difference approximation to

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$$

By Taylor's expansion

$$U_{i+1,j} = U\{(i+1)h, jk\}$$

$$= U(x_{i+h}, t_j)$$

$$= U_{ij} + h\left(\frac{\partial U}{\partial x}\right)_{ij} + \frac{1}{2}h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij} + \frac{1}{6}h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij} + \cdots$$

$$U_{i-1,j} = U\{(i-1)h, jk\}$$

$$= U(x_{i-h}, t_j)$$

$$= U_{ij} - h\left(\frac{\partial U}{\partial x}\right)_{ij} + \frac{1}{2!}h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij} - \frac{1}{6}h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij}$$

$$U_{i,j+1} = U\{ih, (j+1)k\}$$

$$= U(x_i, t_{j+k})$$

$$= U_{ij} + k\left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{1}{2!}k^2\left(\frac{\partial^2 U}{\partial t^2}\right) + \frac{1}{6}k^3\left(\frac{\partial^3 U}{\partial t^3}\right) + \cdots$$

Hence

$$T_{ij} = \frac{U_{i,j+1} - U_{ij}}{k} - \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2}$$

$$= \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij} + \frac{1}{2}k\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} - \frac{1}{12}h^2\left(\frac{\partial^4 U}{\partial x^4}\right) + \frac{1}{6}k^2\left(\frac{\partial^3 U}{\partial t^3}\right) - \frac{1}{360}h^4\left(\frac{\partial^6 U}{\partial x^6}\right) \quad \text{--- (38}$$

But U is the solution of the differential equation

So $$\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right) = 0$$

Therefore the principal part of the local truncation error

$$\left(\frac{1}{2}k\frac{\partial^2 U}{\partial t^2} - \frac{1}{12}h^2\frac{\partial^4 U}{\partial x^4}\right)_{ij}$$

Hence

$$T_{ij} = O(k) + O(h^2)$$

Where $k = rh^2 \qquad 0 < r < \frac{1}{2}$

T_{ij} is $O(k)$ or $O(h^2)$

This error may be further reduced by choosing special value for k/h^2 because equation (38 Can be written as

$$T_{ij} = \frac{1}{12}h^2\left(6\frac{k}{h^2}\frac{\partial^2 U}{\partial t^2} - \frac{\partial^4 U}{\partial x^4}\right) + O(k^2) + O(h^4)$$

By the differential equation

$$\frac{\partial}{\partial t} = \frac{\partial^2}{\partial x^2}$$

So $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}\right)$

assuming that these derivatives exist.

So if we put $\frac{Gk}{h^2} = 1$, the expression is the brackets is then Zero and T_{ij} is

$O(k^2) + O(h^4)$.

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Consistancy and Compatibility

Def: The difference equation is Consistent if the limiting value of the local truncation error is zero as $h \to 0$, $k \to 0$.

Ex: The equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

is approximated at the point (ih, jk) by the difference equation

$$\frac{U_{i,j+1} - U_{ij-1}}{2k} = \frac{U_{i+1,j} - 2\{\theta U_{i,j+1} - (1-\theta) U_{ij} - 1\} + U_{i-1,j}}{h^2} = 0$$

Show that the local truncation error at this point as

$$\frac{k^2}{6}\frac{\partial^3 U}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4} + (2\theta - 1)\frac{2k}{h^2}\frac{\partial U}{\partial t} + \frac{k^2}{h^2}\frac{\partial^2 U}{\partial t^2} + O\left(\frac{k^3}{h^2}, h^4, k^4\right)$$

Discuss the Consistecy of this scheme with partial difference equation When:

i) $k = rh$ and ii) $k = rh^2$

Where r is a postive Constant and θ is Variable parameter.

Sol Expansion of the term $U_{i,j+1}$, $U_{i,j-1}$, $U_{i+1,j}$ and $U_{i-1,j}$ about the point (ih, jk) by

Taylor's series and Substitution

$$T_{ij} = \frac{U_{i,j+1} - U_{i,j-1}}{2k} - \frac{U_{i+1,j} - 2\{\theta U_{i,j+1} + (1-\theta) U_{i,j-1}\} + U_{i-1,j}}{h^2}$$

leads to

$$\overline{T_{ij}} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right) + \frac{k^2}{6}\frac{\partial^3 U}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4} + (2\theta - 1)\frac{\partial U}{\partial t} + \frac{k^2}{h^2}\frac{\partial^2 U}{\partial t^2} + O\left(\frac{k^3}{h^2}, h^4, k^4\right)$$

Case (i) $k = rh$

As $h \to 0$

$$\overline{T_{ij}} = F(U_{ij}) \to \left\{\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + (2\theta - 1)\frac{2r}{h}\frac{\partial U}{\partial t} + r^2\frac{\partial^2 U}{\partial t^2}\right\}_{ij}$$

When $\theta \neq \frac{1}{2}$ the third term to infinity, When $\theta = \frac{1}{2}$ the limiting value of $\overline{T_{ij}}$ is

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + r^2\frac{\partial^2 U}{\partial t^2}$$

In this case the finit difference equation is consistent with the hyperabolic equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + r^2\frac{\partial^2 U}{\partial t^2} = 0$$

Hence the difference equation is inconsistent with

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + r\frac{\partial^2 U}{\partial x^2} = 0 \quad \text{Where} \quad k = rh.$$

$\not\Rightarrow$

Case (ii)

$k = rh^2$

As $h \to 0$

$$T_{ij} \to \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + 2(2\theta - 1) r \frac{\partial U}{\partial t}$$

When $\theta \neq \frac{1}{2}$ the difference scheme is Consistent with the parabolic equation

$$\{1 + 2(2\theta - 1)r\} \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$$

It is only $\theta = \frac{1}{2}$ the difference equation is Consistent with the given differential equation.

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Chapter_4_
Hyperabolic Equation and Characteristic

Finite difference methods on a rectangular mesh for first order equations

Lax_Wendroff explicit method:

Consider

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

Where a is a postive constant.

By Taylors expansions:

$$U_{i,j+1} = U(x_i, t_{j+k}) = U_{ij} + k\left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{1}{2}k^2\left(\frac{\partial^2 U}{\partial t^2}\right) + \cdots$$

Where $x_i = ih$, $t_j = jk$ $\quad i = 0, \pm 1, \pm 2, \cdots$ $\quad j = 0, 1, 2, \cdots$

The differential equation can now be used to eliminate the t-derivative

$$\frac{\partial}{\partial t} = -a\frac{\partial}{\partial x}$$

So

$$U_{i,j+1} = U_{ij} - ka\left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{1}{2}k^2a^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} + \cdots$$

Finally the replacement x-derivatives by central difference approximation gives, to terms in k^2 the explicit equation

$$u_{i,j+1} = u_{ij} - \frac{ka}{2h}(u_{i+1,j} - u_{i-1,j}) + \frac{k^2a^2}{2h^2}(u_{i-1,j} - 2u_{ij} + u_{i+1,j})$$

$$u_{i,j+1} = \frac{1}{2}a\rho(1+a\rho)u_{i-1,j} + (1-a^2\rho^2)u_{ij} - \frac{1}{2}a\rho(1-a\rho)u_{i+1,j} \quad \text{--- (39)}$$

Where $\rho = \frac{k}{h}$.